where u, v, or w is the x, y, or z component of the velocity vector; a is the speed of sound; $\gamma Rs' = s$ is the entropy per unit mass; and subscript x, y, or z denotes partial differentiation with respect to x, y, or z.

As is well known, a characteristic surface allows possible discontinuity of derivatives of flow quantities in its normal direction; in other words, given initial data on such a surface, the normal derivatives of flow quantities are not uniquely determined by the fundamental equations (1–4). Now, let the rectangular coordinates (x, y, z) be so oriented that the x and y axes are tangent to a characteristic surface at the origin, which may be located anywhere in the flow field; then the normal derivative w_z is not uniquely determined by Eqs. (1–4). Solving (1–3)† for w_z in terms of u, v, w, s', and their derivatives with respect to x and y, we obtain $w_z = detN/detD$, where

$$detN = \begin{vmatrix} (a^2 - u^2)u_x + (a^2 - v^2)v_y - uv(v_x + u_y) - w(uw_x + vw_y) & u & v \\ v(u_y - v_x) - ww_x - a^2 s_x' & -1 & 0 \\ u(v_x - u_y) - ww_y - a^2 s_y' & 0 & -1 \end{vmatrix}$$

and

$$detD = \begin{vmatrix} w^2 - a^2 & u & v \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

A necessary and sufficient condition for w_z to be indeterminate is $detD = 0$ and $detN = 0$. The first equation leads to $w^2 - a^2 = 0$, or

$$w = \pm a \tag{5}$$

The second equation yields

$$(w^2 - u^2)u_x + (w^2 - v^2)v_y - uv(v_x + u_y) - 2w(uw_x + vw_y) - a^2(us_x' + vs_y') = 0 \tag{6}$$

Equation (5) states that the velocity component normal to a characteristic surface is equal to the local speed of sound; in other words, a characteristic surface is everywhere tangent to the local Mach cone. Substitution of (5) into (6) yields the corresponding compatibility relation

$$(a^2 - u^2)u_x + (a^2 - v^2)v_y - uv(v_x + u_y) \mp 2a(uw_x + vw_y) - a^2(us_x' + vs_y') = 0 \tag{7}$$

Similarly, solving (2–4) for s_z', we find

$$s_z' = -(us_x' + vs_y')/w$$

The condition that s_z' is indeterminate leads to $w = 0$, $us_x' + vs_y' = 0$. Hence, we see that all surfaces composed of streamlines are also characteristic surfaces and that the corresponding compatibility relation is the constancy of entropy along streamlines. It may be noted that stream surfaces are no longer characteristic when, in addition, the flow is assumed to be irrotational,[3] since s_z' is now identically zero.

When the y axis is chosen to coincide with a generatrix of the Mach cone, the velocity component u vanishes and Eq. (7) assumes the simpler form

$$(a^2 - v^2)v_y + a^2u_x \mp 2avw_y - a^2vs_y' = 0 \qquad \text{or}$$

$$(1 - \cot^2\mu)v_y + u_x \mp 2\cot\mu w_y - (a^2 \cot\mu s_y'/q \sin\mu) = 0 \tag{7'}$$

where μ is the Mach angle. It may be noted that (7′) can be transformed into Ferri's form[1] in the velocity-oriented coordinates (see Appendix).

In conclusion, it is noted that the present approach may be applied to general unsteady three-dimensional flow and, in fact, to hyperbolic equations in n dimensions.

† The same result will be obtained by including (4), which is redundant for finding w_z.

Appendix

We will show that (7′) can be transformed into the forms of other investigators. The rectangular coordinates will first be rotated by an angle $\pm\mu$ with the x axis kept fixed so that the new y' axis coincides with the velocity vector. Let v' and w' denote the velocity components in the new y' and z' direction; the following transformation relations are valid:

$$v_y = \cos^2\mu v_{y'}' \mp \sin\mu\cos\mu(v_{z'}' + w_{y'}') + \sin^2\mu w_{z'}'$$

$$w_y = \cos^2\mu w_{y'}' \mp \sin\mu\cos\mu(w_{z'}' - v_{y'}') - \sin^2\mu v_{z'}'$$

$$s_y' = \cos\mu s_{y'}' \mp \sin\mu s_{z'}' = \mp \sin\mu s_{z'}'$$

since $s_{y'}' = 0$. Substitution of these relations into (7′) yields

$$-\cot\mu v_{y'}' \pm v_{z'}' \mp w_{y'}' + \tan\mu(w_{z'}' + u_x) \pm (a^2/q)s_{z'}' = 0$$

or, with M denoting the Mach number,

$$-(M^2 - 1)v_{y'}' \pm (M^2 - 1)^{1/2}(v_{z'}' - w_{y'}') + w_{z'}' + u_x \pm (a^2/q)(M^2 - 1)^{1/2}s_{z'}' = 0 \tag{8}$$

which is a compatibility relation that can be obtained by straightforward application of the general theory of Ref. 3. Now, Eq. (8) can be reduced to (20-19) of Ref. 1 when it is noted that y', z', and x correspond to t, n, and N, and when Eqs. (20-11, 20-15, and 20-16) of Ref. 1 are used.

References

[1] Ferri, A., "The method of characteristics," *General Theory of High Speed Aerodynamics*, edited by W. R. Sears (Princeton University Press, Princeton, N. J., 1954), p. 583.

[2] Fowell, L. R. and Haskell, R. N., "Memorandum report on the method of characteristics, general theory and description for three-dimensional flow," Convair Memo. Rept. A-1082 (1956).

[3] von Mises, R., *Mathematical Theory of Compressible Fluid Flow* (Academic Press, Inc., New York, 1958), Chap. II, p. 100.

[4] Butler, D. S., "The numerical solution of hyperbolic systems of partial differential equations in three independent variables," Proc. Roy. Soc. (London) **A255**, 232–252 (1960).

[5] Rusanov, V. V., "Characteristics of the general equations of gas dynamics," Zh. Vychislitelnoi Mat. i Mat. Fiz. **3**, 508–527 (1963).

Extension of *f* and *g* Series to Non-Two-Body Forces

Robert M. L. Baker Jr.*
Lockheed California Company, Los Angeles, Calif.

Background

Basic to a number of our orbit determination on prediction routines is the use of the classical f and g series of celestial mechanics. This series, in turn, is based upon an assumption of two-body motion.

In many applications of the f and g series expressions, the accuracy to which the f_i and g_i are developed is far in excess of the accuracy with which the two-body problem truly represents the physical orbit. Such an occurrence poses no difficulty when the f and g series is used for the generation of a reference two-body orbit, but it does pose a problem if it is used indiscriminately to represent observations accurately or to underlie a definitive ephemeris.

Presented as Preprint 64-33 at the AIAA Aerospace Sciences Meeting, New York, January 20–22, 1964.

* Manager, Lockheed Astrodynamics Research Center. Associate Fellow Member AIAA.

Clearly in the case of both near-earth satellite orbits and cislunar trajectories, the two-body assumption is not always reasonable. In the following analyses, the f and g series is extended to a non-two-body form in which influences occasioned, for example, by a spacecraft moving through a known rotating planetary atmosphere, the second harmonic of an aspherical planet, and the influence of a third body are considered.

A generalized form for the f and g expression is proposed as follows:

$$\mathbf{r}_i = r_0 f_i + \dot{\mathbf{r}}_0 g_i + \mathbf{r}_0 \times \dot{\mathbf{r}}_0 h_i$$

where the series takes the form

$$f_i = 1 + \tau_1^2 f_2 + \tau_i^3 f_3 + \ldots$$

$$g_i = \tau_i + \tau_i^2 g_2 + \tau_i^3 g_3 + \ldots$$

$$h_i = h_2 \tau_i^2 + \ldots$$

At least one previous extension of the f and g series has been already proposed. In a 1952 paper by Herrick,[1] we find that he takes into account the perturbing influence of the sun plus the average influence of the four inner planets. When translated into the notation of this paper, we find the f and g series to be formulated as

$$\mathbf{r}_i = f_i \mathbf{r}_0 + g_i \dot{\mathbf{r}}_0 + h_i' \mathbf{r}_{13}$$

Such a formulation has the advantage of placing all of the perturbations in the h_i' term and the disadvantage of generality, i.e., drag, equatorial bulge, more than three bodies, etc., cannot simply be included.

Introduction

An extension of the f and g series to include non-two-body forces can be carried out under the approximation that the perturbative forces exhibit some mean value during the time interval of interest. For the purpose of simplicity, this mean value is taken as the value of perturbative force at the middle date t_0 [i.e., at $\tau = 0$, where $\tau = k(t_i - t_0)$]. A constant mean acceleration $\bar{\ddot{\mathbf{r}}}$ will give rise to a departure from the unperturbed two-body orbit (similar in principle to a reference Encke orbit) amounting to

$$\bar{\ddot{\mathbf{r}}} \tau^2/2 \qquad (1)$$

The bar indicates an average value of the perturbative acceleration, which is ordinarily taken to be the value at time t_2. The extension of the f and g series is, then, simply the resolution of this departure vector along the $\mathbf{r}_0$, $\dot{\mathbf{r}}_0$, and $\mathbf{r}_0, \times \dot{\mathbf{r}}_0$ direction exhibited in the generalized f and g formula:

$$r_i = f_i \mathbf{r}_0 + g_i \mathbf{r}_0 + h_i(\mathbf{r}_0 \times \dot{\mathbf{r}}_0) \qquad (2a)$$

while

$$\dot{\mathbf{r}}_i = \dot{f}_i \mathbf{r}_0 + \dot{g}_i \mathbf{r}_0 + \dot{h}_i(\mathbf{r}_0 \times \dot{\mathbf{r}}_0) \qquad (2b)$$

where the dot term involves $\bar{\ddot{\mathbf{r}}}\tau$ rather than Eq. (1). Our notation will follow:

$$\left.\begin{aligned} f_i &= 1 + \tau_i^2 f_2 + \tau_i^3 f_3 + \ \ldots \\ \dot{f}_i &= 2\tau_i f_2 + 3\tau_i^2 f_3 + \ldots \\ g_i &= \tau_i + \tau_i^2 + g_2 + \tau_i^3 g_3 + \ldots \\ \dot{g}_i &= 1 + 2\tau_i g_2 + 3\tau_i^2 g_3 \ldots \\ h_i &= \tau_i^2 h_2 + \ldots \\ \dot{h}_i &= 2\tau_i h_2 + \ldots \end{aligned}\right\} \qquad (3)$$

so that actually we will be developing only f_2, g_2, and h_2 in what follows.

A possible problem could arise in connection with Eqs. (2a) and (2b) in the case of nearly rectilinear orbits in which $r_0 \times \dot{\mathbf{r}}_0$ becomes nearly indeterminant. In such a case, one could take the value of $\mathbf{r}_0 \times \dot{\mathbf{r}}_0$ as accurate to all of the required figures by definition and adopt the direction $\mathbf{r}_0 \times \dot{\mathbf{r}}_0/|\mathbf{r}_0 \times \dot{\mathbf{r}}_0|$ as fixed for the sake of consistency.

The analysis develops the departure vector due to drag (in a rotating earth atmosphere), due to the aspherical earth, and due to the influence of a third (or more) body. The value of introducing these perturbative influences into the f and g series is only apparent if the mean value approximation adds significantly to the accuracy with which the f and g series represents the true orbit. Because of this, the paper will also present a number of numerical examples, which will demonstrate comparisons among the true orbit (obtained by special perturbations, i.e., Cowell's method), the standard f and g series two-body approximation, and the extended f and g series approximation.

Expressions for f_i, g_i, and h_i

A cancellation of the τ_i^2 terms yields

$$\bar{\ddot{\mathbf{r}}}_0 = \bar{f}_2 \mathbf{r}_0 + \bar{g}_2 \dot{\mathbf{r}}_0 + \bar{h}_2(\mathbf{r}_0 \times \dot{\mathbf{r}}_0) \qquad (4)$$

A dot product of Eq. (4), first with $\mathbf{r}_0$ and then with $\dot{\mathbf{r}}_0$, yields

$$\begin{aligned} \bar{\ddot{\mathbf{r}}}_0 \cdot \mathbf{r}_0/2 &= \bar{f}_2 r_0^2 + \bar{g}_2 r_0 \dot{r}_0 \\ \bar{\ddot{\mathbf{r}}}_0 \cdot \dot{\mathbf{r}}_0/2 &= \bar{f}_2 r_0 \dot{r}_0 + \bar{g}_2 \dot{s}_0^2 \end{aligned} \qquad (5)$$

If the first of these equations is multiplied by $\dot{s}_0^2$, the second by $-r_0\dot{r}_0$, and they are added, then we find that g_2 is eliminated and

$$\begin{aligned} \bar{f}_2 &= \bar{\ddot{\mathbf{r}}}_0 \cdot (r_0 \dot{s}_0^2 - \dot{r}_0 r_0 \dot{r}_0)/2\mu p \\ [r^4 \dot{v}_0^2 &= r_0^2(\dot{s}_0^2 - \dot{\mathbf{r}}_0^2) = \mu p] \end{aligned} \qquad (6)$$

If the first of these equations is multiplied by $\dot{r}_0$, the second by $-r_0$, and they are added, then we find that f_2 is eliminated and

$$\bar{g}_2 = \bar{\ddot{\mathbf{r}}}_0 \cdot (\dot{\mathbf{r}}_0 r_0 - \dot{\mathbf{r}}_0 \dot{r})r_0/2\mu p \qquad (7)$$

A dot product of Eq. (4) by $(\mathbf{r}_0 \times \dot{\mathbf{r}}_0)$ yields

$$\bar{h}_2 = \bar{\ddot{\mathbf{r}}}_0 \cdot (\mathbf{r}_0 \times \dot{\mathbf{r}}_0)/2\mu p \qquad (8)$$

Explicit expressions for $\bar{\ddot{\mathbf{r}}}_0$ can now be introduced into Eqs. (6–8) and values for $\bar{f}_2$, $\bar{g}_2$, and $\bar{h}_2$ thereby obtained.

Influence of Drag

The mean drag perturbation is approximately

$$\bar{\ddot{\mathbf{r}}}_D = -D_0^2 \bar{\sigma} |\bar{\mathbf{v}}| \bar{\mathbf{v}} \qquad (9)$$

where

$D_0^2 = C_{D_0} A_0 \rho_0 V_{c0}^2 / 2 g_0 m_0$
$\bar{\sigma} = \bar{\rho}/\rho_0$
$\bar{v}$ = mean velocity of the spacecraft relative to the resistive medium over the time interval of interest (in characteristic units)
C_{D_0} = reference value of the drag coefficient
A_0 = projected frontal area of the spacecraft, m²
ρ_0 = sea-level atmospheric density, 1.225 kg/m³
V_{c_0} = surface circular satellite speed, 7,905.258 m/sec
g_0 = acceleration of gravity at unit distance, 9.780,320 m/sec²
m_0 = mass of the space vehicle, kg
$\bar{\rho}$ = mean value of atmospheric density over the time interval of interest

Note that the subscript zero will be used interchangeably with 2 to denote the central of expansion point date. Throughout the rest of the paper, we shall approximate the mean value of ν by

$$\bar{\nu} = \dot{\mathbf{r}}_0 - \omega \times r_0 \qquad (10)$$

where ω is the rotational rate of the earth $|\omega| = 0.058273829$ rad/k_e^{-1} min. Solution of Eqs. (6) and (7) for f_i and g_i and of Eq. (8) for h_i yields

$$2f_2/\tau^2 = f_i = \{D_0^2\bar{\sigma}\bar{v}_0 r_0\dot{r}_0\omega(y_0\dot{x}_0 - x_0\dot{y}_0)/2\mu p\}\tau^2 \tag{11a}$$

$$2\bar{g}_2/\tau^2 = g_i = \{-D_0^2\bar{\sigma}\bar{v}_0[1 + \omega r_0^2(y_0\dot{x}_0 - x_0\dot{y}_0)/\mu p]\}\tau^2 \tag{11b}$$

$$2h_2/\tau^2 = h_i = \{D_0^2\bar{\sigma}\bar{v}_0 r_0\omega_0(a_0\dot{r}_0 - r_0\dot{z}_0)/2\mu p\}\tau^2 \tag{11c}$$

The $\dot{f}_i$, $\dot{g}_i$, and $\dot{h}_i$ can be obtained by simply multiplying Eqs. (11) by $2/\tau$.

Influence of the Equatorial Bulge

In the case of the equatorial bulge, one can recognize that the major in-plane perturbative departure (i.e., in the $\mathbf{r}_0 - \dot{\mathbf{r}}_0$ plane) will be essentially a constant factor multiplying μ, i.e., one can consider the local influence at the aspherical earth simply as an augmentation on diminution of the effective earth mass. Clearly, such a change in μ would not cause an out-of-plane perturbation, and so a different tack would be taken in the case of h_i.

In this report, however, f_i, g_i, and h_i are dealt with in the same fashion for reasons of consistency, and μ, although included, may be taken to have a value of unity. The x, y, z components of the average bulge perturbations are taken as

$$\dot{\mathbf{r}}^{\backslash} = \dot{x}^{\backslash}{}_0 I + \dot{y}^{\backslash}{}_0 J + \dot{z}^{\backslash}{}_0 K$$

where

$$\dot{x}^{\backslash}{}_0 = -\frac{k_e^2 x_0}{r_0^3}\left[\frac{3}{2}\frac{J_2}{r_0^2}\left\{1 - 5\left(\frac{z_0}{r_0}\right)^2\right\} + \frac{5}{8}\frac{J_3}{r_0^3}\left\{3 - 7\left(\frac{z_0}{r_0}\right)^2\left(\frac{z_0}{r_0}\right)^4\right\} + \ldots\right]$$

$$\dot{y}^{\backslash}{}_0 = (y_0/x_0)\dot{x}^{\backslash}$$

$$\dot{z}^{\backslash}{}_0 = -\frac{k_e^2 z_0}{r_0^3}\left[\frac{3}{2}\frac{J_2}{r_0^2}\left\{3 - 5\left(\frac{z_0}{r_0}\right)^2\right\} + \frac{5}{2}\frac{J_3}{r_0^3}\left\{6 - 7\left(\frac{z_0}{r_0}\right)^2\right\}\left(\frac{z_0}{r_0}\right) - \frac{5}{8}\frac{J_4}{r_0^4} \times \left\{15 - 70\left(\frac{z_0}{r_0}\right)^2 + 231\left(\frac{z_0}{r_0}\right)^4\right\} + \ldots\right]$$

Three-Body Perturbations

For three (or more) bodies, the average perturbative acceleration has the form

$$\dot{\mathbf{r}}^{\backslash} = m_3\{(\bar{\mathbf{r}}_{23}/r_{23}^3) - (\bar{\mathbf{r}}_{13}/r_{13}^3)\} + \ldots$$

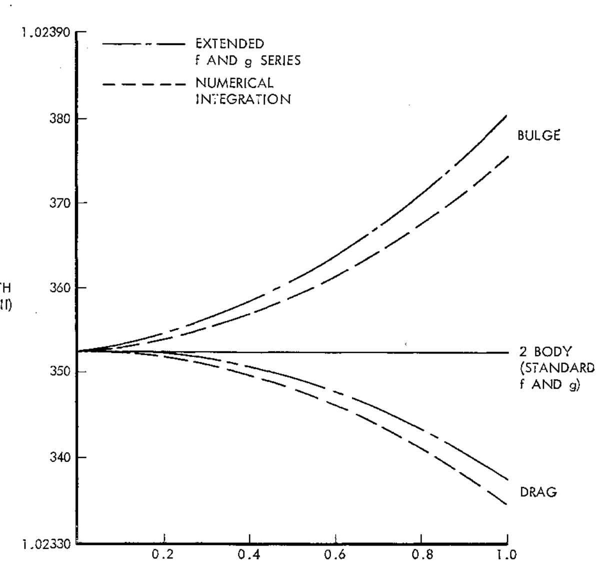

Fig. 1 Case 1, a = 1.023,517,80.

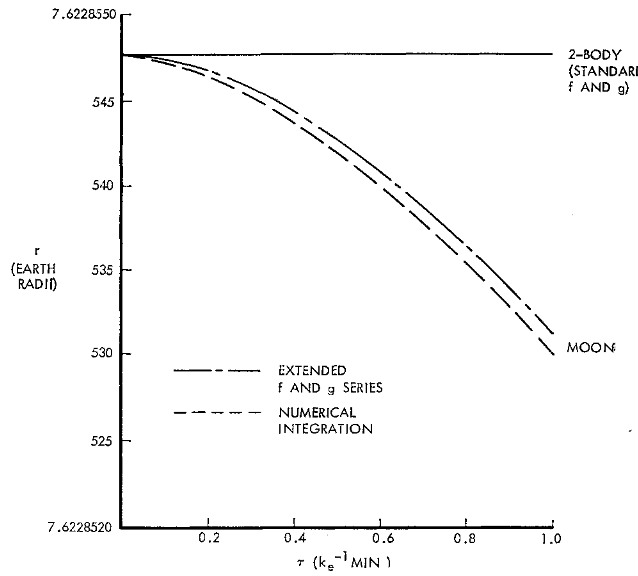

Fig. 2 Case 2, a = 7.622,854,75.

In this case, the $\bar{\mathbf{r}}_{23}$ and $\bar{\mathbf{r}}_{13}$ vectors locate the fixed "mean" position of the pertubing body having a mass m_3.

Numerical Example

Two satellite orbits were chosen for numerical checks, the first with a = 1.023,517,80 earth radii. Both orbits were circular and inclined 45° to the equator. An ARDC 1960 standard atmosphere was used, and the moon's coordinates were taken to be

$\bar{x}_{13}$ = −36.742,345,86 earth radii
$\bar{y}_{13}$ = −36.742,345,86 earth radii
$\bar{z}_{13}$ = +30.0 earth radii

whereas the first body's mean position and velocity were

x_0 = 0.511,758,90 earth radii
$\dot{x}_0$ = −0.698,935 radii/k_e^{-1}
y_0 = 0.511,758,90 earth radii
$\dot{y}_0$ = +0.698,935 radii/k_e^{-1}
z_0 = 0.723,736,37 earth radii
$\dot{z}_0$ = 0.0

and those of the second were

x_0 = 3.811,427,37 earth radii
$\dot{x}_0$ = −0.256,109,77
y_0 = 3.811,427,37 earth radii
$\dot{y}_0$ = −0.256,109,77
z_0 = 5.390,172,27 earth radii
$\dot{z}_0$ = 0.0

Figures 1 and 2 indicate the calculations using a standard two-body f and g series, an extended f and g series, and a Cowell numerical integration to compare the first and second satellite orbit, respectively. Note that the influence of the moon and bulge is negligible for the first orbit, and drag is negligible for the second. (The moon's mass was artificially augmented by a factor of 10 in order to show its influence in a more prominent fashion.)

These figures indicate that after about 0.3 of a k_e^{-1} min (about 4 min) an additional significant figure can be gained through use of the extension of the f and g series, and after about 0.5 of a k_e^{-1} min two or more significant figures can be added through its use.

Reference

[1] Herrick, S., "Jupiter IX and XII," Publ. Astron. Soc. Pacific 64, 237–241 (October 1952); also Univ. of California at Los Angeles, Astronomical Paper 1, pp. 131–138.